
FURTHER MATHEMATICS

9231/21

Paper 2

October/November 2017

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Mark Scheme Notes

Marks are of the following three types:

- M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.
- A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).
- B Mark for a correct result or statement independent of method marks.
- When a part of a question has two or more “method” steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.
 - The symbol FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously “correct” answers or results obtained from incorrect working.
 - Note: B2 or A2 means that the candidate can earn 2 or 0.
B2/1/0 means that the candidate can earn anything from 0 to 2.

The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.
- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no “follow through” from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous’ answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become “follow through” marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

| Question | Answer | Marks | Guidance |
|----------|--|--------------|--|
| 1 | $-p + 2t = 0, p = 4$ | M1 A1 | Find p by equating transverse acceln. to 0 at $t = 2$ |
| | $a_R = (8 - pt + t^2)^2 / 0.8$ | M1 | Find radial acceleration a_R at $t = 2$ in terms of p from v^2/r |
| | $= 4^2 / 0.8 = 20 \text{ [m s}^{-2}\text{]}$ | A1 | Evaluate with $p = 4$ |
| | | 4 | |

| Question | Answer | Marks | Guidance |
|----------|--|--------------|--|
| 2 | $(3\pi/5)^2 = \omega^2 (a^2 - (a - 1.6)^2)$ (AEF) | M1 A1 | Use $v^2 = \omega^2 (a^2 - x^2)$ in each posn. (M1 for either) |
| | $(\pi/4)^2 = \omega^2 (a^2 - (a - 0.2)^2)$ $(3\pi/5)^2 (0.4a - 0.04) = (\pi/4)^2 (3.2a - 2.56)$ | A1 | Combine to find amplitude a and ω^2 (or ω) |
| | $a = 2.6 \text{ [m]}$ | M1 A1 | (M1 for either) |
| | $\omega^2 = (\pi/4)^2$ or $\omega = \pi/4$ (AEF) | A1 | Find other unknown |
| | $T = 2\pi / (\pi/4) = 8 \text{ [s]}$ | B1FT | Find period T from $T = 2\pi/\omega$ (\surd on ω^2 or ω) |
| | | 7 | |

| Question | Answer | Marks | Guidance |
|----------|--|----------|---|
| 3(i) | $mv_A + kmv_B = 2mu + kmu$ [$v_A + kv_B = 2u + ku$] | (AEF) | M1 Use conservation of momentum for <i>A</i> & <i>B</i> (allow omission of <i>m</i> in all momentum eqns) |
| | $v_B - v_A = \frac{1}{2}(2u - u)$ [= $\frac{1}{2}u$] | | M1 Use Newton's restitution law with consistent LHS signs |
| | $v_B = u(2k + 5) / 2(k + 1)$ or $u(k + 5/2) / (k + 1)$ | (AEF) | A1 Combine to find v_B |
| | [$v_A = u(k + 4) / 2(k + 1)$] $v_B > 4u/3$ if $k < 7/2$ | | M1 A1 Find inequality for <i>k</i> from speeds of <i>B</i> and <i>C</i> after 1st collision |
| | | | 5 |
| 3(ii) | $kmw_B + mv_C = kmv_B + m(4u/3)$ [$2w_B + v_C = 2(3u/2) + 4u/3$ = $13u/3$ when $k = 2$] | (AEF) | M1 Use conservation of momentum for <i>B</i> & <i>C</i> |
| | $v_C - w_B = \frac{1}{2}(v_B - 4u/3)$ [= $u/12$] $(k + 1)w_B = (k - \frac{1}{2})v_B + 2u$ | | M1 Use Newton's restitution law with consistent LHS signs Combine to find w_B |
| | $3w_B = (3/2)v_B + 2u$ with $v_B = 3u/2$, so $w_B = 17u/12$ | | *A1 when $k = 2$ |
| | $v_A = u, v_A < w_B$ | | DB1 Verify no further collisions between <i>A</i> and <i>B</i> |
| | <i>EITHER:</i> $(k + 1)v_C = (3k/2)v_B + (2 - k)(2u/3)$ $3v_C = 3v_B$ with $v_B = 3u/2$ so $v_C = 3u/2 > w_B$ | | (DB1) <i>EITHER:</i> Find v_C and verify no further collisions between <i>B</i> and <i>C</i> |
| | <i>OR:</i> <i>B</i> and <i>C</i> cannot meet again since they move apart after colliding (AEF) | | (DB1) <i>OR:</i> State explicitly that no further collisions between <i>B</i> and <i>C</i> |
| | | 5 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|---|
| 4(i) | $T \times 2a = (5W/2) \times a \cos \theta$ | M1 | Take moments for rod about A |
| | $T = \frac{1}{2} (5W/2) \times (4/5) = W$ | A1 | to find tension T |
| | | 2 | |
| 4(ii) | $T \sin \theta = \mu (W + T \cos \theta)$ | M1 | Use $F_P = \mu R_P$ at P |
| | $(3/5) W = \mu (1 + 4/5)W = \mu (9/5)W, \mu = 1/3$ | A1 | to find μ |
| | | 2 | |
| 4(iii) | <i>EITHER:</i> $[\pm] X = T \sin \theta = 3W/5$ or $0.6W$ $[\pm] Y = 5W/2 - T \cos \theta$ or $5W/2 + W - R_P$ | (B1) | <i>EITHER:</i> Find horizontal component X of force at A Find vertical component Y of force at A |
| | $= 17W/10$ or $1.7W$ | (B1) | |
| | <i>OR:</i> $[\pm] X = (5W/2) \sin \theta = 3W/2$ or $1.5W$ | (B1) | <i>OR:</i> Find component X of force at A along BA |
| | $[\pm] Y = (5W/2) \cos \theta - T = W$ | (B1) | Find component Y of force at A perp. to BA |
| | $R_A^2 = X^2 + Y^2 = 13W^2/4, R_A = \frac{1}{2}W\sqrt{13}$ or $1.80 W$ | B1FT | Find magnitude of resultant force R_A at A (FT on X, Y) |
| | | 3 | |
| 4(iv) | $PB = (a + 2a \sin \theta) / \cos \theta = 5a/4 + 3a/2 = 11a/4$ or $2.75a$ or $x = 3a/4$ or $0.75a$ | B1 | Find length PB or extension x of string |
| | $T = \lambda (PB - 2a)/2a, \lambda = 8W/3$ or $2.67W$ | M1 A1 | Find modulus λ using Hooke's Law |
| | | 3 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|--|
| 5(i) | $A_{ABCD} = 24a^2$, $A_{EFGH} = 8a^2$, $A_{Frame} = 16a^2$ $M_{ABCD} = 3m/2$ and $m_{EFGH} = m/2$ | B1 | Use areas to find masses M_{ABCD} and m_{EFGH} |
| | $I_{ABCD} = \frac{1}{3} M_{ABCD} ((3a)^2 + (2a)^2)$ [= (13/3) $M_{ABCD} a^2 = (13/2) ma^2$] | B1 | Find MI of $ABCD$ about axis at centre Q |
| | $I_{EFGH} = \frac{1}{3} m_{EFGH} ((2a)^2 + a^2)$ [= (5/3) $m_{EFGH} a^2 = (5/6) ma^2$] | B1 | Find MI of $EFGH$ about axis at centre Q |
| | $I_{ABCD} - I_{EFGH} + m \times (6a)^2$ [= (121/2 - 113/6) ma^2 or (17/3 + 36) $ma^2 = (125/3) ma^2$] | M1 A1 | Find MI of frame about axis at O Result also follows from combining four rectangular parts |
| | $I_{Object} = (11m/12) \times (4a)^2$ [= (44/3) ma^2] | B1 | Find MI of small object about axis at O |
| | $I = (13/2 - 5/6 + 36 + 44/3) ma^2 = (169/3) ma^2$ AG | A1 | Combine to verify MI of system about axis at O |
| | | 7 | |
| 5(ii) | $[-] I d^2\theta/dt^2$ or $[-] I\alpha$ = $mg \times 6a \sin \theta + (11/12)mg \times 4a \sin \theta$ or $(23/12)mg \times (116/23)a \sin \theta$ [= (29/3) $mga \sin \theta$] | M1 A1 | Use eqn of circular motion to relate $d^2\theta/dt^2$ to $\sin \theta$, where θ is angle of QO with vertical |
| | $d^2\theta/dt^2$ or $\alpha = -(29g / 169a) \theta$ or $-(0.172 g / a) \theta$ | M1 A1 | Approximate $\sin \theta$ by θ to show SHM (M0 if wrong sign or $\cos \theta \approx \theta$ used) |
| | $T = 2\pi / \sqrt{(29g/169a)}$ = $26\pi \sqrt{(a/29g)}$ or $15.2\sqrt{(a/g)}$ or $4.80\sqrt{a}$ (AEF) | A1 | Find period T from $T = 2\pi/\omega$ (A1 requires some simplification) |
| | | 5 | |

| Question | Answer | Marks | Guidance |
|----------|------------------------------------|--------------|---|
| 6(i) | $p = (1/6)^2$ or $1/36$ | B1 | Find (or imply) probability p of pair of sixes in one throw |
| | $1/p = 36$ | B1 | Find mean value of X |
| 6(ii) | $P(X = 12) = p(1-p)^{11} = 0.0204$ | M1 A1 | Find prob. of needing exactly 12 throws |
| 6(iii) | $P(X > 12) = (1-p)^{12} = 0.713$ | M1 A1 | Find prob. of needing more than 12 throws |
| | | 2 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|--|
| 7(i) | $F(x) = \int f(x) dx = -e^{-0.2x} + c = 1 - e^{-0.2x} (x \geq 0)$ | M1 | State, or integrate and use $F(0) = 0$ or $F(x) \rightarrow 1$ as $x \rightarrow \infty$ |
| | and $F(x) = 0 (x < 0$ or otherwise) | A1 | to find, $F(x)$ (A0 if case $x < 0$ omitted) |
| | | 2 | |
| 7(ii) | $P(X > 2) = 1 - F(2) = e^{-0.4} = 0.670$ | M1 A1 | Find $P(X > 2)$: (M0 for $F(2)$) |
| | | 3 | |
| 7(iii) | $1 - e^{-0.2m} = 1/2, e^{0.2m} = 2$ | M1 | Find median value m from $F(m)$ or $1 - F(m) = 1/2$ |
| | $m = 5 \ln 2$ or 3.47 | M1 A1 | |
| | | 3 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|---|
| 8 | H_0 : No association or method is independent of gender (AEF) | B1 | State (at least) null hypothesis |
| | E_i : 9.0 19.8 31.2 6.0 13.2 20.8 | M1 A1 | Find expected values E_i (A0 if rounded to integers) |
| | $\chi^2 = 0.111 + 3.073 + 1.482$ $+ 0.167 + 4.610 + 2.223$ $= 11.7$ (or 12.3) (to 2 d.p.) | M1 A1 | Find value of χ^2 from $\Sigma (E_i - O_i)^2 / E_i$ [or $\Sigma O_i^2 / E_i - n$] (allow 12.3 if integer values of E_i used) |
| | $\chi_{2, 0.99}^2 = 9.21$ | B1 | State or use correct tabular χ^2 value |
| | Reject H_0 if $\chi^2 >$ tabular value (AEF) | M1 | Valid method for reaching conclusion |
| | 11.7 [± 0.1] $>$ 9.21 so there is an association | A1 | Correct conclusion, from correct values |
| | | 8 | |

| Question | Answer | Marks | Guidance |
|----------|--|--------------|---|
| 9(i) | $r = S_{xy} / \sqrt{(S_{xx}S_{yy})}$ with e.g. $S_{xy} = 212.62 - 35.9 \times 36.8/8 = 47.48$ (or 5.935) $S_{xx} = 216.47 - 35.9^2/8 = 55.37$ (or 6.921) $S_{yy} = 244.96 - 36.8^2/8 = 75.68$ (or 9.46) (all to 3 s.f.) | M1 A1 | Find correlation coefficient r (Insufficient working loses first A1) |
| | $r = 0.733$ | *A1 | |
| | | 3 | |

| Question | Answer | Marks | Guidance |
|----------|--|---------------|--|
| 9(ii) | $H_0: \rho = 0, H_1: \rho > 0$ | B1 | State both hypotheses (B0 for $r \dots$) |
| | <i>EITHER:</i> $r_{8, 1\%} = 0.789$ | (*B1) | State or use correct tabular one-tail r -value |
| | Accept H_0 if $ r < \text{tab. } r\text{-value}$ (AEF) | M1) | State or imply valid method for conclusion |
| | <i>OR:</i> $t_r = r\sqrt{(n-2) / (1 - r^2)} = 2.64, t_{6, 0.99} = 3.143$ | (*B1) | |
| | Accept H_0 if $ t_r < \text{tab. } t\text{-value}$ (AEF) | M1) | |
| | No positive correlation (AEF) | DA1) | |
| | | 4 | |
| 9(iii) | $r_{14, 1\%} = 0.661, r_{15, 1\%} = 0.641$ so $n_{min} = 15$ | M1 A1) | Find n_{min} from relevant two-tail tabular value[s] SC: Award B1 for stating 15 without any justification |
| | | 2 | |

| Question | Answer | Marks | Guidance |
|----------|--|------------------------|---|
| 10(i) | $\bar{x} = 0.91, \bar{y} = 1.205$ $s_x^2 = 19.56 / 49 [= 489/1225 \text{ or } 0.3992]$ and $s_y^2 = 30.25 / 59 [= 121/236 \text{ or } 0.5127]$ | B1 M1 | Find both sample means Estimate both population variances (allow biased here: 0.3912 and 0.5042) |
| | <i>EITHER:</i> $s^2 = s_x^2/50 + s_y^2/60$ | (M1) | Estimate or imply combined variance |
| | $= 0.01653 \text{ or } 0.1286^2$ (to 3 s.f. throughout) | A1 | |
| | $[\pm] (\bar{y} - \bar{x}) \pm z s$ | M1 | Find confidence interval for difference $Y - X$ or $X - Y$ |
| | $z_{0.95} = 1.645$ $[\pm] 0.295 \pm 0.211$ (allow 0.212) | A1 | Use appropriate tabular value Evaluate confidence interval (either form) |
| | or $[\pm] [0.084, 0.506]$ (allow $[\pm] [0.083, 0.507]$) | A1) | |
| | <i>OR:</i> Assume equal [population] variances $s^2 = (49 s_x^2 + 59 s_y^2) / 108$ or $(19.56 + 30.25) / 108$ | (B1) | State assumption Find or imply pooled estimate of common variance (note s_x^2 and s_y^2 not needed explicitly so first M1 may be implied by result) |
| | $= 4981/10800 \text{ or } 0.461 \text{ or } 0.679^2$ | B1 | |
| | $[\pm] (\bar{y} - \bar{x}) \pm z s \sqrt{(1/50 + 1/60)}$ | M1 | Find confidence interval for difference $Y - X$ or $X - Y$ |
| | $z_{0.95} = 1.645$ | A1 | Use appropriate tabular z -value (or t -value from calculator) |
| | $[\pm] 0.295 \pm 0.214$ or $[\pm] [0.081, 0.509]$ | A1) | Evaluate confidence interval (either form) |
| | 7 | | |

| Question | Answer | Marks | Guidance |
|----------|--|--------------|---|
| 10(ii) | $z = (1.205 - 0.91) / s = 0.295 / s$ $= 2.29[4] \quad [or\ 2.26[9]]$ (to 3 s.f.) | M1 A1 | Find value of z (either sign) |
| | $(z) = 0.989[1] \quad [or\ 0.988[4]]$ | A1 | Find $\Phi(z)$ |
| | $100 \times (1 - 0.989) \times 2 = 2.2 \quad [or\ 2.3]$ (to 1 d.p.) | M1 A1 | Find limiting value for α , based on two-tail test (M0 for basing on one-tail test) |
| | $\alpha < (or \leq) 2.2 \quad [or\ 2.3]$ | A1 | Find set of possible values of α (Treat α instead of $\alpha\%$ as misread) |
| | | 6 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|---|
| 11A(i) | $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mga (\cos \alpha + \cos \beta)$ | M1 A1 | Find v^2 at A' from conservation of energy (A0 if no m) |
| | $mv^2/a = mg \cos \beta$ $u^2 = ag \cos \beta + 2ag (1/16 + \cos \beta)$ | B1 | Use $F = ma$ radially at A' with $R_{A'} = 0$ Use $\cos \alpha = 1/16$ and eliminate v^2 to verify u^2 |
| | $= (1/8) ag (1 + 24 \cos \beta)$ AG | M1 A1 | |
| | | 5 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|--|
| 11A(ii) | $\frac{1}{2}mw^2 = \frac{1}{2}mu^2 - mga(1 + \cos 2\beta)$ | B1 | Find w^2 at B' from conservation of energy (B0 if no m) |
| | $mw^2/a = mg \cos 2\beta$ $u^2 = ag \cos 2\beta + 2ag(1 + \cos 2\beta)$ | B1 | Use $F = ma$ radially at B' with $R_{B'} = 0$ Eliminate w^2 to find u^2 |
| | $= ag(2 + 3 \cos 2\beta)$ | M1 | |
| | $1 + 24 \cos \beta = 8(2 + 3 \cos 2\beta)$ $= 16 + 48 \cos^2 \beta - 24$ $16 \cos^2 \beta - 8 \cos \beta - 3 = 0$ | M1 | Combine eqns for u^2 Formulate and solve quadratic to find $\cos \beta$ |
| | $\cos \beta = \frac{3}{4}$ [rejecting $-\frac{1}{4}$] | M1 A1 | |
| | | 6 | |
| 11A(iii) | $u^2 = (19/8) ag$ or $2.375 ag$ $R = mu^2/a + mg \cos \alpha$ $= (19/8 + 1/16) mg$ | B1 | Find u^2 using value of $\cos \beta$ Use $F = ma$ radially at A to find reaction R at A |
| | $= (39/16) mg$ or $2.44 mg$ | M1 A1 | |
| | | 3 | |

| Question | Answer | Marks | Guidance |
|----------|---|----------------|--|
| 11B(i) | e.g. $S_{xy} = 218.72 - 45.3 \times 40.5/9 = 14.87$ or 1.652 $S_{xx} = 245.59 - 45.3^2/9 = 17.58$ or 1.953 or $S_{yy} = 195.11 - 40.5^2/9 = 12.86$ or 1.429 | | Find reqd. values |
| | <i>EITHER:</i> $b_1 = S_{xy} / S_{xx} = 14.87 / 17.58 = 0.84585 = 0.846$ $(y - 40.5/9) = b_1 (x - 45.3/9)$ $(y - 4.5) = 0.846 (x - 5.033)$ | (M1 A1) | Find gradient in $y - \bar{y} = b_1 (x - \bar{x})$ to 3 s.f. Find eqn. of regression line to 3 s.f. |
| | $y = 0.846x + 0.24257 = 0.846x + 0.243$ | M1 A1 | |
| | $x = 4.68$ | A1) | Find x when $y = 4.2$ |
| | <i>OR:</i> $b_2 = S_{xy} / S_{yy} = 14.87 / 12.86 = 1.1563 = 1.16$ $(x - 45.3/9) = b_2 (y - 40.5/9)$ $(x - 5.033) = 1.16 (y - 4.5)$ | (M1 A1) | Find gradient in $x - \bar{x} = b_2 (y - \bar{y})$ to 3 s.f. Find eqn. of regression line to 3 s.f. |
| | $x = 1.16y - 0.170[04]$ | M1A1 | |
| | $x = 4.69$ | A1) | Find x when $y = 4.2$ (A0 for $x = 4.70$) |
| | | 5 | |

| Question | Answer | Marks | Guidance |
|----------|---|--------------|---|
| 11B(ii) | $H_0: \mu_x - \mu_y = 0.3, H_1: \mu_x - \mu_y > 0.3$ (AEF) | B1 | State hypotheses (B0 for $\bar{x} \dots$) |
| | $\bar{d} = 4.8 / 9$ or $8/15$ or 0.533 (where $d = x - y$) | B1 | Find sample mean |
| | $s^2 = (3.26 - 4.8^2/9) / 8$ $= 7/80$ or 0.0875 or 0.296^2 | M1 A1 | Estimate population variance (allow biased here: $7/90$ or 0.0778 or 0.279^2) |
| | $t = (\bar{d} - 0.3) / (s/\sqrt{9}) = 2.37$ | M1 A1 | Find value of t |
| | $t_{8, 0.975} = 2.306$ or 2.31 | B1 | State or use correct tabular t -value (or can compare \bar{d} with $0.3 + t_{8, 0.975} s/\sqrt{9} = 0.527$) |
| | Reject H_0 if $t >$ tabular value (AEF) $2.37 [\pm 0.1] > 2.31$ so accept belief | M1 | Valid method for reaching conclusion Correct conclusion, from correct values |
| | [of increase of more than 0.3] (AEF) | A1 | SC: Wrong (hypothesis) test can earn only B1 for hypotheses |
| | | | 9 |