RECOGNISING ACHIEVEMENT

GCE

Mathematics

Advanced GCE **4727**

Further Pure Mathematics 3

Mark Scheme for June 2010

Oxford Cambridge and RSA Examinations

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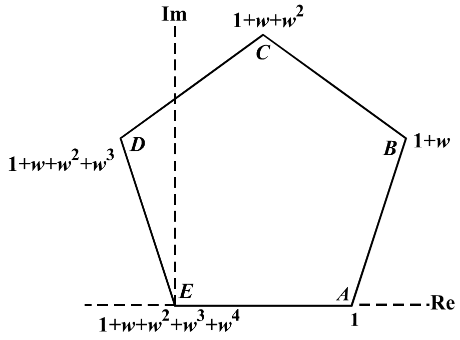
1	$\left.\begin{array}{l}\text{Direction of } l_1 = k[7, 0, -10] \\ \text{Direction of } l_2 = k[1, 3, -1]\end{array}\right\}$	B1	For both directions
	EITHER $\mathbf{n} = [7, 0, -10] \times [1, 3, -1]$ $OR \left\{\begin{array}{l}[x, y, z] \,.\, [7, 0, -10] = 0 \;\Rightarrow\; 7x - 10z = 0 \\ [x, y, z] \,.\, [1, 3, -1] = 0 \;\Rightarrow\; x + 3y - z = 0\end{array}\right.$	M1	For finding vector product of directions of l_1 and l_2 *OR* for using 2 scalar products and obtaining equations
	$\Rightarrow \mathbf{n} = k[10, -1, 7]$	A1	For correct **n**
	METHOD 1 Vector $(\mathbf{a}-\mathbf{b})$ from l_1 to $l_2 = \pm[4, 6, -10]$ *OR* $\pm[-4, 3, 1]$ *OR* $\pm[3, 3, -9]$ *OR* $\pm[-3, 6, 0]$	B1	For a correct vector
	$d = \dfrac{\lvert(\mathbf{a}-\mathbf{b}) \,.\, \mathbf{n}\rvert}{\lvert\mathbf{n}\rvert} = \dfrac{36}{\sqrt{150}}$	M1*	For finding $(\mathbf{a}-\mathbf{b}) \,.\, \mathbf{n}$
		M1 (*dep)	For $\lvert\mathbf{n}\rvert$ in denominator *OR* for using $\hat{\mathbf{n}}$
	$d = \frac{6}{5}\sqrt{6} \approx 2.94$	A1 7	For correct distance **AEF**
	METHOD 2 Planes containing l_1 and l_2 perp. to **n**	M1*	For finding planes and $p_1 - p_2$ seen
	are $\mathbf{r} \,.\, [10, -1, 7] = p_1 = 70$, $\mathbf{r} \,.\, [10, -1, 7] = p_2 = 34$	B1	For $p_1 = 70k$ and $p_2 = 34k$
	$\Rightarrow d = \dfrac{\lvert 70 - 34\rvert}{\sqrt{150}} = \dfrac{36}{\sqrt{150}} = \frac{6}{5}\sqrt{6} \approx 2.94$	M1 (*dep)	For $\lvert\mathbf{n}\rvert$ in denominator *OR* for using $\hat{\mathbf{n}}$
		A1	For correct distance **AEF**
	METHOD 3 $\mathbf{r}_1 = [7\lambda, 0, 10 - 10\lambda]$ *OR* $[7 + 7\lambda, 0, -10\lambda]$ $\mathbf{r}_2 = [4 + \mu, 6 + 3\mu, -\mu]$ *OR* $[3 + \mu, 3 + 3\mu, 1 - \mu]$	B1	For correct points on l_1 and l_2 using different parameters
	$\begin{array}{rcccccc} 7\lambda + 10\alpha - \mu & = & \vert\; 4 & \vert\; -3 & \vert\; 3 & \vert\; -4 & \vert \\ -\alpha - 3\mu & = & \vert\; 6 & \vert\; 6 & \vert\; 3 & \vert\; 3 & \vert \\ -10\lambda + 7\alpha + \mu & = & \vert\; -10 & \vert\; 0 & \vert\; -9 & \vert\; 1 & \vert \end{array}$ $\Rightarrow \alpha = -\frac{6}{25}$	M1*	For setting up 3 linear equations from $\mathbf{r}_1 + \alpha\mathbf{n} = \mathbf{r}_2$ and solving for α
	$\lvert\mathbf{n}\rvert = \sqrt{150}$	M1 (*dep)	For $\lvert\mathbf{n}\rvert$ seen multiplying α
	$\Rightarrow d = \frac{6}{25}\sqrt{150} = \frac{6}{5}\sqrt{6} \approx 2.94$	A1	For correct distance **AEF**
		$\boxed{7}$	

2	**(i)**	$ar = r^5a \Rightarrow rar = r^6a$	M1		Pre-multiply $ar = r^5a$ by r
		$r^6 = e \Rightarrow rar = a$	A1	**2**	Use $r^6 = e$ and obtain answer **AG**
	(ii)	METHOD 1			
		For $n = 1$, $rar = a$ *OR* For $n = 0$, $r^0ar^0 = a$	B1		For stating true for $n = 1$ *OR* for $n = 0$
		Assume $r^kar^k = a$			
		EITHER Assumption $\Rightarrow$ $r^{k+1}ar^{k+1} = rar = a$	M1		For attempt to prove true for $k + 1$
		OR $r^{k+1}ar^{k+1} = r.r^kar^k.r = rar = a$			
		OR $r^{k+1}ar^{k+1} = r^k.rar.r^k = r^kar^k = a$	A1		For obtaining correct form
		Hence true for all $n \in \mathbb{Z}^+$	A1	**4**	For statement of induction conclusion
		METHOD 2			
		$r^2ar^2 = r.rar.r = rar = a$, similarly for $r^3ar^3 = a$	M1		For attempt to prove for $n = 2, 3$
		$r^4ar^4 = r.r^3ar^3.r = rar = a$, similarly for $r^5ar^5 = a$	A1		For proving true for $n = 2, 3, 4, 5$
		$r^6ar^6 = eae = a$	B1		For showing true for $n = 6$
		For $n > 6$, $r^n = r^{n \bmod 6}$, hence true for all $n \in \mathbb{Z}^+$	A1		For using n mod 6 and correct conclusion
		METHOD 3			
		$r^nar^n = r^{n-1}.rar.r^{n-1}$ *OR* $r^nar^n = r^n.r^5a.r^{n-1} = r^{n+5}ar^{n-1}$	M1		Starting from n, for attempt to prove true for $n-1$
		$= r^{n-1}ar^{n-1}$	A1		For proving true for $n-1$
		$= r^{n-2}ar^{n-2} = \ldots$	A1		For continuation from $n-2$ downwards
		$= rar = a$	B1		For final use of $rar = a$ **SR** can be done in reverse
		METHOD 4			
		$ar = r^5a \Rightarrow ar^2 = r^5ar = r^{10}a$ etc.	M1		For attempt to derive $ar^n = r^{5n}a$
		$\Rightarrow ar^n = r^{5n}a$	A1		For correct equation **SR** may be stated without proof
		$\Rightarrow r^nar^n = r^{6n}a$	B1		For pre-multiplication by r^n
		$= ea = a$	A1		For obtaining a ($r^6 = e$ may be implied)
				$\boxed{6}$	

3					Allow $\text{cis}\frac{k}{5}\pi$ and $e^{\frac{k}{5}\pi i}$ throughout
	(i)	$w^2=\cos\frac{4}{5}\pi+i\sin\frac{4}{5}\pi$	B1		For correct value
		$w^3=\cos\frac{6}{5}\pi+i\sin\frac{6}{5}\pi$	B1		For correct value
		$w^*=\cos\frac{2}{5}\pi-i\sin\frac{2}{5}\pi$	B1		For w^* seen or implied
		$=\cos\frac{8}{5}\pi+i\sin\frac{8}{5}\pi$	B1	4	For correct value **SR** For exponential form with i missing, award B0 first time, allow others
	(ii)		B1* B1 (*dep) B1 (*dep) B1	4	For $1+w$ in approximately correct position For $AB\approx BC\approx CD$ For BC, CD equally inclined to Im axis For E at the origin Allow points joined by arcs, or not joined Labels not essential
	(iii)	$z^5-1=0$ *OR* $z^5+z^4+z^3+z^2+z=0$	B1	1	For correct equation **AEF** (in any variable) Allow factorised forms using w, exp or trig
			$\boxed{9}$		
4	**(i)**	$y=xz\Rightarrow\frac{dy}{dx}=z+x\frac{dz}{dx}$	B1		For correct differentiation of substitution
		$\Rightarrow xz+x^2\frac{dz}{dx}-xz=x\cos z\Rightarrow x\frac{dz}{dx}=\cos z$	M1 A1		For substituting into DE For DE in variables separable form
		$\Rightarrow\int\sec z\,dz=\int\frac{1}{x}dx$	M1		For attempt at integration to ln form on LHS
		$\Rightarrow\ln(\sec z+\tan z)=\ln kx$ *OR* $\ln\tan\left(\frac{1}{2}z+\frac{1}{4}\pi\right)=\ln kx$	A1		For correct integration (k not required here)
		$\Rightarrow\sec\left(\frac{y}{x}\right)+\tan\left(\frac{y}{x}\right)=kx$ *OR* $\tan\left(\frac{y}{2x}+\frac{1}{4}\pi\right)=kx$	A1	6	For correct solution **AEF** including $\text{RHS}=e^{(\ln x)+c}$
	(ii)	$(4,\pi)\Rightarrow\sec\frac{1}{4}\pi+\tan\frac{1}{4}\pi=4k$ *OR* $\tan\left(\frac{1}{8}\pi+\frac{1}{4}\pi\right)=4k$	M1		For substituting $(4,\pi)$ into their solution (with k)
		$\Rightarrow\sec\left(\frac{y}{x}\right)+\tan\left(\frac{y}{x}\right)=\frac{1}{4}\left(1+\sqrt{2}\right)x$ *OR* $\tan\left(\frac{y}{2x}+\frac{1}{4}\pi\right)=\left(\frac{1}{4}\tan\frac{3}{8}\pi\right)x$ *or* $\frac{1}{4}\left(1+\sqrt{2}\right)x$	A1	2	For correct solution **AEF** Allow decimal equivalent $0.60355x$ Allow $e^{\ln x}$ for x
			$\boxed{8}$		

5	**(i)**	$C+\mathrm{i}S=1+\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}+\frac{1}{4}\mathrm{e}^{2\mathrm{i}\theta}+\frac{1}{8}\mathrm{e}^{3\mathrm{i}\theta}+\ldots$	M1		For using $\cos n\theta+\mathrm{i}\sin n\theta=\mathrm{e}^{\mathrm{i}n\theta}$ at least once for $n\geqslant 2$
			A1		For correct series
		$=\dfrac{1}{1-\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}=\dfrac{2}{2-\mathrm{e}^{\mathrm{i}\theta}}$	M1		For using sum of infinite GP
			A1	**4**	For correct expression **AG** **SR** For omission of 1st stage award up to M0 A0 M1 A1 **OEW**
	(ii)	$C+\mathrm{i}S=\dfrac{2\left(2-\mathrm{e}^{-\mathrm{i}\theta}\right)}{\left(2-\mathrm{e}^{\mathrm{i}\theta}\right)\left(2-\mathrm{e}^{-\mathrm{i}\theta}\right)}$	M1		For multiplying top and bottom by complex conjugate
		$=\dfrac{4-2\mathrm{e}^{-\mathrm{i}\theta}}{4-2\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+1}=\dfrac{4-2\cos\theta+2\mathrm{i}\sin\theta}{4-4\cos\theta+1}$	M1		For reverting to cosθ and sinθ and equating Re *OR* Im parts
		$\Rightarrow C=\dfrac{4-2\cos\theta}{5-4\cos\theta},\quad S=\dfrac{2\sin\theta}{5-4\cos\theta}$	A1		For correct expression for C **AG**
			A1	**4**	For correct expression for S
				$\boxed{8}$	
6	**(i)**	Aux. equation $m^2+2m+17(=0)$	M1		For attempting to solve correct auxiliary equation
		$\Rightarrow m=-1\pm 4\mathrm{i}$	A1		For correct roots
		CF $(y=)\ \mathrm{e}^{-x}\left(A\cos 4x+B\sin 4x\right)$	A1√		For correct CF (allow $A\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(4x+\varepsilon)$) (trig terms required, not $\mathrm{e}^{\pm 4\mathrm{i}x}$) f.t. from their m with 2 arbitrary constants
		PI $(y=)\ px+q\ \Rightarrow\ 2p+17(px+q)=17x+36$	M1		For stating and substituting PI of correct form
		$\Rightarrow\ p=1$	A1		For correct value of p
		and $q=2$	A1		For correct value of q
		GS $y=\mathrm{e}^{-x}\left(A\cos 4x+B\sin 4x\right)+x+2$	B1√	**7**	For GS. f.t. from their CF+PI with 2 arbitrary constants in CF and none in PI. Requires $\boxed{y=}$.
	(ii)	$x\gg 0\Rightarrow \mathrm{e}^{-x}\to 0$ *OR* very small	B1		For correct statement. Allow graph
		$\Rightarrow y=x+2$ approximately	B1√	**2**	For correct equation Allow ≈ , → and in words Allow relevant f.t. from linear part of GS
				$\boxed{9}$	

7	**(i)**	$(1, 3, 5)$ and $(5, 2, 5) \Rightarrow \pm[4, -1, 0]$ in Π	M1		For finding a vector in Π
		$\mathbf{n} = [2, -2, 3] \times [4, -1, 0] = k[1, 4, 2]$	M1		For finding vector product of direction vectors of l and a line in Π
			A1		For correct **n**
		$\Rightarrow \mathbf{r}.[1, 4, 2] = 23$	A1	4	For correct equation. Allow multiples
	(ii)	METHOD 1			
		Perpendicular to Π through $(-7, -3, 0)$ meets Π	M1		For using perpendicular from point on l to Π Award mark for $k\mathbf{n}$ used
		where $(-7+k)+4(-3+4k)+2(2k)=23$	M1		For substituting parametric line coords into Π
		$\Rightarrow k=2 \quad \Rightarrow d = 2\sqrt{1^2+4^2+2^2} = 2\sqrt{21} \approx 9.165$	M1		For normalising the **n** used in this part
			A1	4	For correct distance **AEF**
		METHOD 2			
		Π is $x+4y+2z=23$	M1		For attempt to use formula for perpendicular distance
		$\Rightarrow d = \dfrac{\lvert(-7)+4(-3)+2(0)-23\rvert}{\sqrt{1^2+4^2+2^2}} = 2\sqrt{21} \approx 9.165$	M1		For substituting a point on l into plane equation
			M1		For normalising the **n** used in this part
			A1		For correct distance **AEF**
		METHOD 3			
		$\mathbf{m} = [1, 3, 5] - [-7, -3, 0] = (\pm)[8, 6, 5]$	M1		For finding a vector from l to Π
		OR $= [5, 2, 5] - [-7, -3, 0] = (\pm)[12, 5, 5]$			
		$\Rightarrow d = \dfrac{\mathbf{m}.[1, 4, 2]}{\sqrt{1^2+4^2+2^2}} = \dfrac{42}{\sqrt{21}} = 2\sqrt{21} \approx 9.165$	M1		For finding $\mathbf{m} . \mathbf{n}$
			M1		For normalising the **n** used in this part
			A1		For correct distance **AEF**
		METHOD 4			As Method 1, using parametric form of Π
		$[-7, -3, 0] + k[1, 4, 2] = [1, 3, 5] + s[2, -2, 3] + t[4, -1, 0]$	M1		For using perpendicular from point on l to Π Award mark for $k\mathbf{n}$ used
		$\left.\begin{array}{ll} k-2s-4t & =8 \\ 4k+2s+t & =6 \\ 2k-3s & =5 \end{array}\right\} \Rightarrow k=2 \quad \left(s=-\frac{1}{3},\ t=-\frac{4}{3}\right)$	M1		For setting up and solving 3 equations
		$\Rightarrow d = 2\sqrt{1^2+4^2+2^2} = 2\sqrt{21} \approx 9.165$	M1		For normalising the **n** used in this part
			A1		For correct distance **AEF**
		METHOD 5			
		$d_1 = \dfrac{23}{\sqrt{1^2+4^2+2^2}} = \dfrac{23}{\sqrt{21}}$	M1		For attempt to find distance from O to Π *OR* from O to parallel plane containing l
		$d_2 = \dfrac{[-7, -3, 0].[1, 4, 2]}{\sqrt{1^2+4^2+2^2}} = \dfrac{-19}{\sqrt{21}}$	M1		For normalising the **n** used in this part
		$\Rightarrow d_1 - d_2 = d = \dfrac{23-(-19)}{\sqrt{21}} = 2\sqrt{21} \approx 9.165$	M1		For finding $d_1 - d_2$
			A1		For correct distance **AEF**
	(iii)	$(-7, -3, 0) + k(1, 4, 2)$	M1		State or imply coordinates of a point on the reflected line
		Use $k=4$	M1		State or imply 2 × distance from **(ii)** Allow $k = \pm 4$ *OR* $\pm 4\sqrt{21}$ f.t. from **(ii)**
		$\mathbf{b} = [2, -2, 3]$	B1		For stating correct direction
		$\mathbf{a} = [-3, 13, 8]$	A1	4	For correct point seen in equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$
		$\mathbf{r} = [-3, 13, 8] + t[2, -2, 3]$			**AEF** in this form
				$\boxed{12}$	

8	**(i)**	$\{A, D\}$ *OR* $\{A, E\}$ *OR* $\{A, F\}$	B1	1	For stating any one subgroup
	(ii)	*A* is the identity 5 is not a factor of 6 *OR* elements can be only of order 1, 2, 3, 6	B1 B1	 2	For identifying *A* as the identity For reference to factors of 6
	(iii)	$BE = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D, \quad EB = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} = F$ $D \text{ or} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ F \text{ or} \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \in M$ ⇒ closure property satisfied	M1 A1 A1 A1	 4	For finding *BE* and *EB* AND using $\omega^3 = 1$ For correct *BE* (*D or* matrix) For correct *EB* (*F or* matrix) For justifying closure
	(iv)	$B^{-1} = \frac{1}{1}\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} = C$ $E^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -\omega^2 \\ -\omega & 0 \end{pmatrix} = E$	M1 A1 A1	 3	For correct method of finding either inverse For correct $B^{-1} = C$ Allow $\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$ For correct $E^{-1} = E$ Allow $\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$
	(v)	METHOD 1 *M* is not commutative e.g. from $BE \neq EB$ in part **(iii)** *N* is commutative (as × mod 9 is commutative) ⇒ *M* and *N* not isomorphic	 B1 B1 B1#	 3	 For justification of *M* being not commutative For statement that *N* is commutative For correct conclusion
		METHOD 2 Elements of *M* have orders 1, 3, 3, 2, 2, 2 Elements of *N* have orders 1, 6, 3, 2, 3, 6 Different orders *OR* self-inverse elements ⇒ *M* and *N* not isomorphic	B1* B1 (*dep) B1#		For all orders of one group correct For sufficient orders of the other group correct For correct conclusion **SR** Award up to B1 B1 B1 if the self-inverse elements are sufficiently well identified for the groups to be non-isomorphic
		METHOD 3 *M* has no generator since there is no element of order 6 *N* has 2 *OR* 5 as a generator ⇒ *M* and *N* not isomorphic	 B1 B1 B1#		 For all orders of *M* shown correctly For stating that *N* has generator 2 *OR* 5 For correct conclusion
		METHOD 4 $\begin{array}{c\|cccccc} M & A & B & C & D & E & F \\ \hline A & A & B & C & D & E & F \\ B & B & C & A & F & D & E \\ C & C & A & B & E & F & D \\ D & D & E & F & A & B & C \\ E & E & F & D & C & A & B \\ F & F & D & E & B & C & A \end{array}$	B1*		For stating correctly all 6 squared elements of one group
		$\begin{array}{c\|cccccc} N & 1 & 2 & 4 & 8 & 7 & 5 \\ \hline 1 & 1 & 2 & 4 & 8 & 7 & 5 \\ 2 & 2 & 4 & 8 & 7 & 5 & 1 \\ 4 & 4 & 8 & 7 & 5 & 1 & 2 \\ 8 & 8 & 7 & 5 & 1 & 2 & 4 \\ 7 & 7 & 5 & 1 & 2 & 4 & 8 \\ 5 & 5 & 1 & 2 & 4 & 8 & 7 \end{array}$	B1 (*dep)		For stating correctly sufficient squared elements of the other group
		⇒ *M* and *N* not isomorphic	B1#		For correct conclusion
					# In all Methods, the last B1 is dependent on at least one preceding B1
			$\boxed{13}$		

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